

*Faculty of Education and
Languages and Learning Centre
University of Western Sydney*

Unravelling the Mystery of Mathematics

***A Language and Inquiry Approach
to Number and Algebra***

Kay Owens

Unravelling the Mystery of Mathematics:

A language and inquiry approach to number and algebra

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Preface

This booklet was developed to assist students who lack confidence in their mathematics but need to pursue it as part of their professional studies. Many of these students seem to have a stronger interest in the language areas and to enjoy stories and historical ideas. Many of these students found mathematics at school meaningless and boring as they ploughed through endless exercises. For this reason, an approach has been taken which minimises the number of exercises and encourages thinking about different points. However, there are thinking tasks which should not be skipped. These tasks will involve you in thinking mathematically as well as in language. Through thinking mathematically, you are becoming a competent mathematician.

It is a good idea to discuss your thinking with a fellow student and to discuss issues with your tutor or lecturer if you are unsure of the points being raised. If possible, raise the issues with your tutorial group and come to a consensus.

There are suggested answers for the thinking tasks at the back of the book but these should only be consulted when you have been unable to discuss your ideas with others. They are there to take away your anxiety rather than to tell you what the right answers are. If you treat them as answers you should get, you are still trying to learn another person's mathematics rather than developing your own mathematical thinking. And you can do that thinking!

Chapter 1

My Personal Unravelling

1.1 A Metaphor

The cicadas were prolific and loud during the early summer. My brothers and I enjoyed the excitement of exploring a rambling house and garden during our Blue Mountains holiday. We climbed the dense firs and native gums looking for cicadas.

In the crispness of the morning, we found another cicada shell but this tiny airy monster was just beginning to split. In a sliver of sunlight, the little creature inside arched its back and pulled out one leg and then another from its fragile case. The little, soft, moist crumples on its back began to unravel until they became firm, long, and shiny. After years of development, it metamorphosed in a short time.

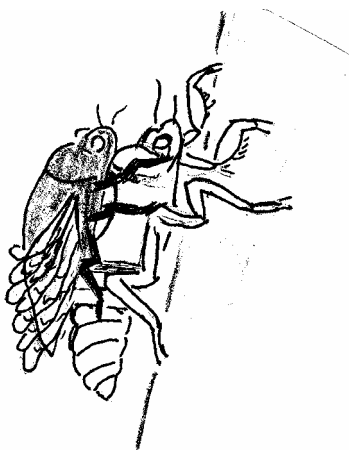


Figure 1.1. A cicada unravelling itself as it leaves the cocoon

My personal journey through mathematics has been much like that morning metamorphosis. At school, I gained ticks for quickly writing down the multiplication facts but I could not tell you the meaning of multiplication. Come to think of it we hardly ever *spoke* mathematics except to answer some direct questions from the teacher as she worked through an example on the board. I could apply numbers to formulae but word problems were always frustrating as the words and mathematical operations just did not relate.

I kept studying mathematics though it became more mysterious and harder to remember. I was searching for a purpose and some meaning. At last at university, I learnt about sets; multiplication and other number work began to fall into place. The fact that equal groups of a number (e.g., 3 groups of 5) could be represented as multiplication was a metamorphic experience allowing many aspects of mathematics to fall into place.

1.2 Changes in My Understanding of Mathematics

My real belief in the value of mathematics began with those sudden delightful new insights that came when pondering on a piece of mathematics. (For a discussion of metamorphic accommodation, see Owens & Clements, 1998 or Steffe, 1988).

Later I taught mathematics to a range of technology degree students, giving me the opportunity to see how mathematics related to the real world.

My greatest metamorphoses and insights came as I grappled with new mathematics for teaching or with my students' discussions. I read books on mathematics that were not standard textbooks. The names of the more memorable books tell how I, and their authors, perceive mathematics. *Freedom to Learn* (Biggs, 1969), *The Lore of Large Numbers* (Davis, 1961), *Mathematics in a Cultural Context* (Harris, 1991), *The Crest of a Peacock* (Joseph, 1992), *On the Shoulders of Giants* (Steen, 1990), *Geometry: An Investigative Approach* (O'Daffer & Clemens, 1977), *Signifying Nothing: The Semiotics of Zero* (Rotman, 1987), *Counting Systems of Papua New Guinea and Oceania* (Lean, 1993), and *Celebrating Women in Mathematics and Science* (Cooney, 1996). My interest in people in mathematics began to grow, and I found mathematics in books like *A Story of Architecture* (Nuttgens, 1983), and on the Italian works of Brunelleschi (Klotz, 1990), and the Swedish Vasa Museum. I hope this book will inspire as these books inspired me.

My Papua New Guinea students helped me recognise the cultural bases of mathematics while my struggling students enjoyed sharing their new understandings. In coming to grips with their faltering explanations, new insights were gained. One of these new appreciations was about place value which I will share in a later chapter.

For a time, I was working with Support Teachers Learning Difficulties exploring why their students were struggling with multiplication and other number problems. Frequently, the difficulties, though not recognised at first, included place value. It is a concept that takes years to develop fully and it requires visual as well as verbal understanding. It is closely linked to an understanding of zero and consistent spatial and number patterns.

The importance of patterns and relationships in primary school mathematics is often espoused. Some claim it is the basis of algebra (for a discussion on these issues see Macgregor and Quinlan, 1996). Patterns are abstract and difficult to see. How can we develop facility with pattern recognition and generalisation. Is it essential? What is its role in thinking mathematically? We need to grapple with such issues.

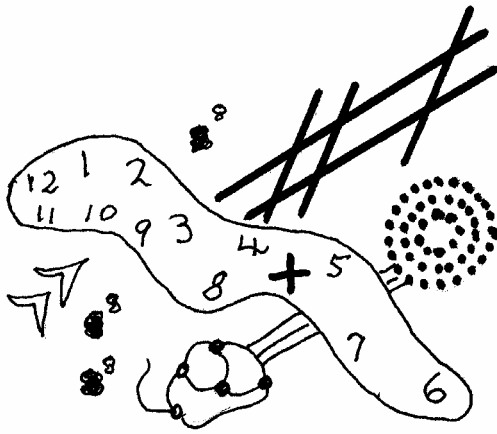


Figure 1.2. A free thought on symbols based on areas of mathematics, Aboriginal and artists' symbols.

1.3 Why Symbols?

I recall my visit to an Arthur Boyd exhibition which explained the inclusion of symbols in his paintings. I have noticed that time symbols and open drawers are integral to Dahl's imagery. Happily we grapple with the symbolism of an artwork or an author's words, but we may not realise the full import of a mathematical symbolic

relationship. Mathematics is full of symbols, and unravelling the richness of these symbols will stand us in good stead in reading and writing mathematics and understanding the concepts behind these symbolic statements.

This book aims at stating what is often a hidden meaning behind a mathematical statement. I hope it will shed new light on an area of mathematics that may have been a meaningless frustration in the past.

Thinking Task 1.1

When have you been excited by a new understanding in mathematics?

Record your thoughts and share them with others. Let's at least try to start with a positive thought.

We will explore some ways in which we can learn well in mathematics. Begin your puzzling with a brainstorm.

Thinking Task 1.2

When and how are lines used in mathematics?

Are there any connections between the different uses you have recorded?

We shall return to these ideas in later chapters but first we will look at another key idea in mathematics that is related to the ideas of symbols but also to the ideas about learning.

Chapter 2

Representation

2.1 A Simple Example of Visual, Symbolic, and Word Representations of Number Relations

This chapter is a mix of shape, number, and algebra in order to illustrate the important of representation and being able to read representations mathematically. If a student can do this, much of the mystery of mathematics disappears, and the beauty of its interconnectedness and elegance can develop.

We can represent mathematical ideas on paper or with physical materials. A line on a page may represent a set of points. We also represent ideas mentally but a physical representation can be a check on our own mental representation. A word problem is often best understood by sketching a diagram, acting it out, or representing it with materials.

Representations in our minds can be either visual or verbal, and frequently both are needed. Indeed, we often develop our initial encounters with ideas into visual imagery, and then we develop conceptual ideas to extend our understandings (Pirie & Kieren, 1991). Realising this, can improve approaches to learning.

If primary school students have difficulties remembering number combinations, then simple 10-frames can be useful to encourage them to picture combinations of 10, and hence combinations beyond 10 (see Figure 2.1).

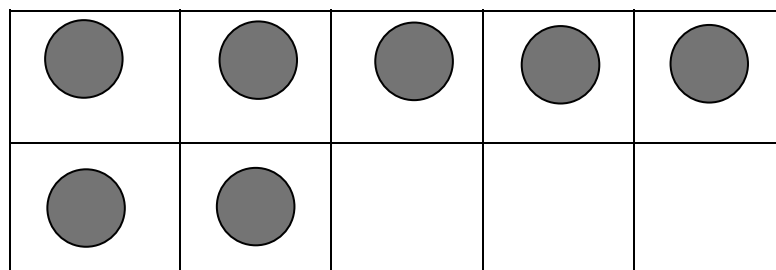


Figure 2.1. A ten frame used to represent number relations.

Thinking Task 2.1

Write all the whole number relationships represented by Figure 2.1. For example,

$$5 + 2 = 7$$

$$10 - 3 = 7$$

Visual representations are surprisingly valuable in mathematics. For example, they assist in mental computations. Adding up from 7 to 10 is 3. So $7 + 4$ is $10 + 1$; that is, a mental image of one beyond 10. Encouraging students to do mental calculations will facilitate the use of “up-to-ten” combinations and other effective strategies.

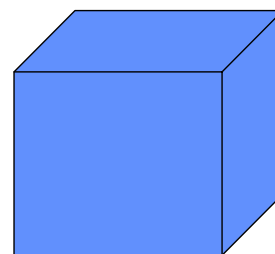
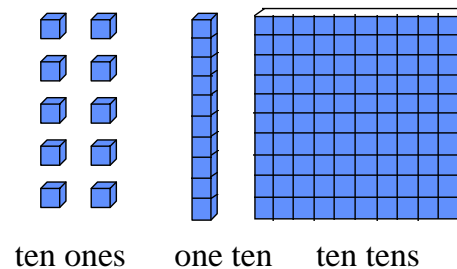
Use an “up-to-ten” strategy to add $7 + 6$. Use another kind of mental strategy. Give other examples

2.2 Sharing Internal Representations with Others

A teacher can draw a scalene triangle on the board as representative of triangles without discussing why a scalene triangle is drawn. The full import of the diagram is lost. The teacher should explain that this triangle was drawn so a special situation (right-angled or equilateral or isosceles triangles) does not interfere with either a visual or mathematical proof.

Base 10 materials such as Dienes blocks might contain meaningful representations about the base 10 place value system but this representation may not be appreciated by students. It must be discussed.

Nevertheless, concrete materials such as base 10 materials are useful to check students’ understanding of numbers.



one thousand

Thinking Task 2.2

Represent the number 1 048 using Dienes base 10 materials

Represent 1 048 with lengths such as metre sticks where 100 is represented by a metre stick.

- *Are you surprised at how long your metre-sticks line is and what small portion of the line is representing 48?*
- *What does it suggest about the size of numbers?*

- *What are the advantages and disadvantages in using each type of concrete material?*

Thinking Task 2.3

Here is another problem. Try using diagrams to answer it. The problem is deliberately vague so look for many different solutions and then try generalising.

A farmer ties his goat to a post on the edge of his shed.

What grass area will he be able to eat?

- *Do you have many different diagrams?*
- *Are you using one diagram to represent a series of possible solutions?*
- *What kind of lines and shapes are used?*
- *Did you give a generalisation for a result?*

Chapter 3

Lines

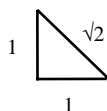
3.1 Number Lines

A commonly used line for number is the *number line*. What are some important features of this line?

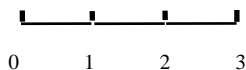
Thinking Task 3.1

- What does a numeral signify?
- Which is the length representing 3?
- How could you get school students to appreciate the idea that it is not just position on the line that represents size but also a length representing size?
- How are numbers compared using the line?
- Why is spacing important?
- Are there scales that use other than equal spacings? Why?
- How can scales help in representing large numbers or numbers to 2 or 3 decimal places?
- How is the number line used for operations (additions, subtractions, multiplications)?
- Do points on the number line represent whole numbers, negative numbers, fractions, irrational numbers like $\sqrt{2}$?
- The last number, $\sqrt{2}$, can be positioned by making a careful diagram of a right-angled triangle of one unit by one unit (based on the unit of the

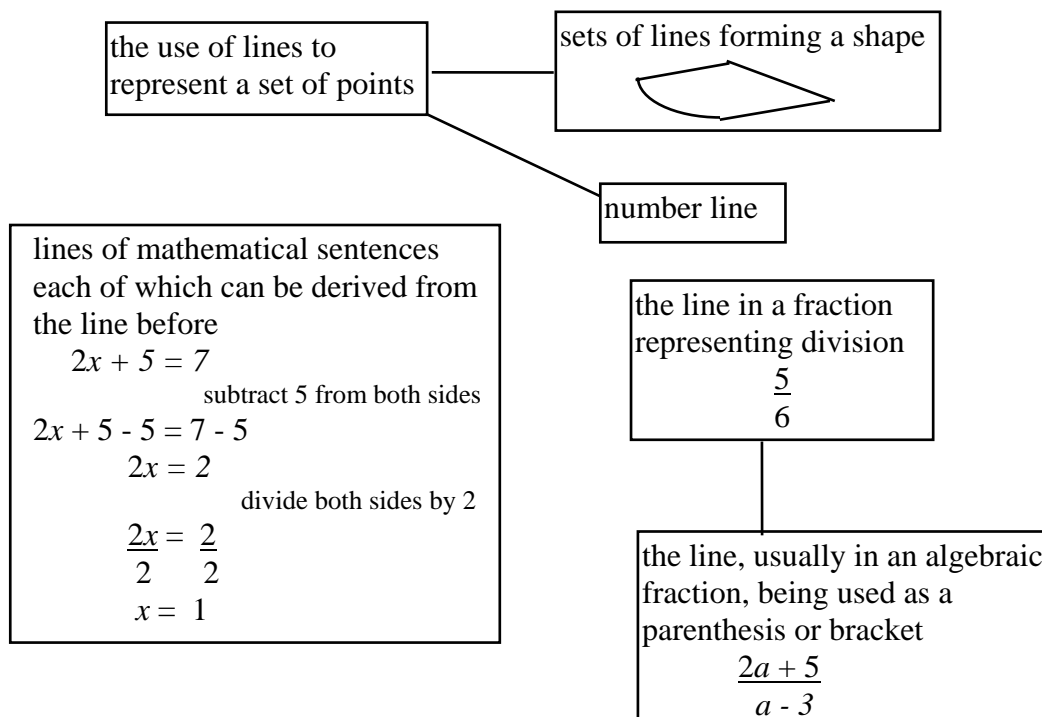
number line). The hypotenuse is $\sqrt{2}$ using Pythagoras' theorem. Use a pair of compasses as dividers to mark the position of $\sqrt{2}$ on the number line.



Pythagoras' theorem
 $a^2 + b^2 = c^2$
 $1^2 + 1^2 = 2$
 $c = \sqrt{2}$



Did your brainstorm on lines in chapter one include:



3.2 Equation Lines

The use of letters in mathematics is perhaps the most confusing of all aspects of mathematics for students. There seems to be no purpose and no easy solutions. This matter will be looked at later but it is first important to be able to understand mathematical reading rules. When you read a book written in English, you begin at the top left corner, read to the right, start on the left of the next line, and read line by line down the page, from the left page to the right page which you then turn over. This is convention. There are conventions for reading mathematics.

Take the following sentences and note that the left side of each line equals the right side. However, the left side of the first line does not equal either side of the lines below.

$$\begin{aligned}2x^2 &= 18 \\x^2 &= 9 \\x &= 3 \text{ or } -3\end{aligned}$$

Each line can be *derived* from the line or sentence above but no part is *equal* to a part of the line above. Each line can be obtained from the previous line. For example, dividing both sides of the first line provides the second line.

We know that squaring both 3 and -3 will give 9 and if we double it we get 18. In other words, the first sentence is true when the last line is true. x is a *pronumeral* that can stand for any number just like we use pronouns (e.g. *he*, *she*) to stand for a person. However the equality is true, in this case, for specific values of x .

In the last paragraph, I have used many words that are often used in mathematics like *equal*, *value*, *pronumeral*. Each is an important idea so check the use and meaning of each.

The equations above have one term equalling another term. Do you understand the use of the word *term*? For example, $2x^2$ is a term.

How does a term differ from an expression in mathematics? An expression may consist of more than one term. Terms can be added or subtracted in an expression. For example, both $2x^2$ and $5x^2 + 3$ are expressions.

In equations we have one expression equalling another expression. For example, $5x^2 + 3 = 7$.

Thinking Task 3.2

How does each line of the following equation derive from the previous line? For example,

$$\begin{aligned}3(x + 5) &= 18 \text{ dividing both sides by } 3 \\(x + 5) &= 6 \\x + 5 &= 6 \\x + 5 - 5 &= 6 - 5 \\x &= 1\end{aligned}$$

Is there another way to solve this equation by removing parentheses first? Try it. Do you get the same answer? You should.

3.3 Expression Lines

However, in mathematics we do not work just with equations. Sometimes we write lines of expressions that are derived from the line above. These are *equal* in one way or another. For example, it may be a simpler way of writing it or it may be the expression with a specific value replacing the pronumeral. Here are two examples:

$$\begin{aligned} & 3 \times 4 + 5 \times 4 \\ & = 12 + 20 \\ & = 32 \end{aligned}$$

$$\begin{aligned} & 3x^2 + 4x \text{ if } x = 5 \\ & = 3 \times 5^2 + 4 \times 5 \\ & = 3 \times 25 + 20 \\ & = 75 + 20 \\ & = 95 \end{aligned}$$

Each line is equal to the line above. The equal sign on the left side signifies this. There were no equal signs on the left side of lines of equations because the lines of equations are not equal, but derived from each other.

Thinking Task 3.3

How do you write lines equal to this line in simplifying the following expression? Remember that 4 multiplies the parenthesis so, by convention, the multiplication sign can be left out.

$$4 (5 - 3)$$

Now try the following. Remember that both terms in the parentheses are multiplied by $2x$. The last line might not look simpler but the parentheses have been removed.

$$2x (3x - 5)$$

3.4 The Fraction Line

This line is also called vinculum which is a Latin word so the idea has been around for a long time. It was derived from ancient Egyptian

hyroglyphics which represented unit fractions with an elongated ellipse
 e.g. $\overline{\text{III}}$ represents the fraction one-third which we write as $\frac{1}{3}$

Now this fraction line has multiply uses. One is to indicate that the number below it gives the number of equal parts that the whole has been divided into. Another use of the line is to indicate that we are referring to a number that has a name like tenths.

Another use of this vinculum (fraction line) is that of a parenthesis or bracket (one of the points made at the start of the chapter). Numbers inside a parenthesis must be kept together unless operated upon conventionally.

Here are some examples:

$$\begin{aligned} & 3 \times \overline{7-5} \\ = & 3 \times 2 \\ = & 6 \end{aligned}$$

$$\begin{aligned} & 3 \times \overline{7-5} \\ = & 3 \times 7 - 3 \times 5 \\ = & 21 - 15 \\ = & 6 \end{aligned}$$

$$\begin{aligned} & \frac{12+5}{7-3} \\ = & \frac{17}{4} \end{aligned}$$

$$\begin{aligned} & \frac{2a+3}{4} = 6 \\ & \frac{2a+3}{4} \times 4 = 6 \times 4 \\ & 2a+3 = 24 \\ & 2a+3-3 = 24-3 \\ & 2a = 21 \\ & a = \frac{21}{2} \end{aligned}$$

Old textbooks often used the vinculum line like the first two examples for parentheses instead of () or []. Remember that operations in parentheses are carried out before others.

In the right-hand example, lines of equations are written. Whatever has been done to one side has been done to the other in order to keep a balance. While expressions under each other are not equal, each line is balanced or the two expressions on the line are equal.

Whenever you see an *algebraic fraction* with a vinculum (line), then it is indicating

- a fraction,
- a division operation, *and*
- a parenthesis or bracket.

Frequently the meaning of a symbol or a representation is not fully appreciated. The line in an algebraic fraction is not only representing division but also a parenthesis or bracket. This is why the denominator subtraction needs to be calculated before dividing. Some students are tempted to wrongly “cancel” the pronumeral first, ignoring the fact that the pronumerals are in parentheses.

Lines of poetry hold messages of beauty if we can interpret them. Lines in mathematics hold meaning too. Mathematics is an amazingly succinct short-hand that holds the power:

- to use one symbol for many different purpose (e.g. the vinculum),
- to be useful to help in rows of calculations so we can keep track,
- to provide ways of applying simple operations like addition to complex problems in algebra,
- to be able to apply consistently in many different types of problems, the same tactics like “undoing” an operation by using the opposite operation to that being represented, e.g. adding to both sides of an equation, when there is a subtraction.

Thinking Task 3.4

Draw up a diagram to represent your understanding of **number**. You might like to use overlapping circles, columns dividing into more columns, or branching trees. Share your ideas with another person.

Chapter 4

Numbers and Place Value

4.1 Different Kinds of Numbers

Did your diagram of numbers include all the different types of numbers such as counting numbers, zero, fractions, decimal fraction numbers, negative numbers, as well as place value, base 10 system, square roots, π ?

There are other ideas such as

- rational numbers (e.g., fractions), and irrational numbers (e.g. $\sqrt{3}$ or π),
- logarithms or the power of a number relative to a base (e.g., 8 has a power or log of 3 relative to the base of 2 because $2^3 = 8$),
- approximations,
- accuracy to a particular number of significant figures (e.g., 3.5, 2400 are rounded to 2 significant figures)
- numeral representation (e.g., in 12 the numeral 1 has a place value of 10)
- number systems
- number laws

4.2 Number Systems

The symbols we see for numbers again hide much meaning. If we see 2 we think of it referring to two objects or a group with the number attribute of 2. We do not know what number system it is a member of unless we see it with other numbers or we are specifically told. There is even a mathematical shorthand for doing this. For example, $n \in R$ suggests the numbers we are currently dealing with *belong to* (\in) the *set of rational numbers* (R).

However, 2 might be a counting number, a whole number, a rational number, a number in the base 10 system, and a number with place value.

4.3 The Origins of Number Systems

Number systems are developed by societies because different societies have different ways with numbers. Numbers are not a universal truth given to us by nature although their development was influenced by nature (e.g. a hand).

In order to appreciate our number system, it often helps to know about other people's systems.

Austronesian counting systems. Many coastal Melanesian people in the western Pacific have number cycles of 5 (often designated by the same word as that used for a hand) and 20 (having counted digits on 2 hands and 2 feet and sometimes using the word for whole man). Other groups have a dominance of the 10 cycle. Clearly the availability of hands and feet digits influenced the counting strategies of these societies.

Roman numerals. Several different symbols are used for different numbers. Numbers are represented as additions of these.

For example, CCLXIV is two hundred (C), 50 (L is fifty), ten (X), four (V is five so I before V is four), that is 264.

There are two significant features of the system. Different symbols stand for different types of main numbers, and the symbol is listed as many times as needed for those main numbers e.g, CC for two hundred. Second, the position order is important in so far as a symbol before a higher number indicates to take it away e.g., IV for four, or XL for forty, XC for ninety. There was no zero and no easy way of carrying out calculations with these symbols.

Mayan numerals. This ancient middle-Americas society reached a peak of fame around 600AD. Its numeral systems had a cycle of 20 and they had a symbol for zero.

Non-Austronesian Melanesian systems. There are language groups that have cycles based on twos (usually associated with 5, 10, or 20), and threes, fours, fives, sixs, and eights.

Ndom is one of three 6-cycle languages spoken on Kolopom Island adjacent to the south coast of West Papua (Irian Jaya), just west of the PNG border. It is interestingly a Non-Austronesian language on an island. When speakers of Ndom say the number for 7, they use 6 and then add 1, so $7=6+1$. Then when they say 8, they use 6 again and add on 2, so $8=6+2$. 6 becomes a composite unit just like 10 is in our base system. What will 9 be?

When they say 12, the cycle of 6 is complete again and they say $12=2 \times 6$. The cycle continues $13=2 \times 6+1$. What will 14 be?

When they reach 18, they introduce a word "tondor" beginning a super cycle in addition to the 6 cycle. Then $20=18+2$

Ndom has a (6, 18, 36) cyclic pattern. There are distinct terms for 18 and 36. Both 72 and 108 are compounds of 36, that is, $72 = 36 \times 2$ and $108 = 36 \times 3$.

Ndom

1	sas	11	mer abo meregh
2	thef	12	mer an thef
3	ithin	13	mer an thef abo sas

4	thonith	18	tondor
5	meregh	20	tondor-abo-thef
6	mer	24	tondor abo mer
7	mer abo sas	36	nif
8	mer abo thef	40	nif-abo-thinith
9	mer abo ithin	72	nif thef
10	mer abo thonith	108	nif ithin

Sources: Galis (1960, p. 148) & Drabbe (1949a, pp. 6-7) (Cited in Lean, 1992, Vol. 5, p. 122; 1992, pp. 113-114)

Below are the counting words for a Papua New Guinea language, **Melpa**, (data collected before 1941, cited in Lean, 1994). Find the relationships between numbers that include the same word e.g. *ki*. There are cycles in the counting. Can you describe them? They regroup when they get to 8 (*ki*) with a remnant of cycles of 4 (*tembokaka*) and 2 (*rakl*). (The number one is signified by *dende* and *ti*). Nevertheless 10 was predominant in counting in pig ceremonies with 9 and 10 represented by the thumbs.

Melpa

1	dende = 1	13	tembokakapoket pombin ti ngkutl = $3 \times 4 + 1$
2	rakl = 2	14	tembokakapoket pombin rakl ngkutl $3 \times 4 + 2$
3	raktika = $3 = 2 + 1$	15	tembokakapoket pombin raktika ngkutl
4	tembokaka = 4	16	ki rakl = two 8s
5	pombingkutl = $4 + 1$	17	ki rakl tende ngkutl = $2 \times 8 + 1$
6	pombingkutl dende or ngkutl rakl	18	ki rakl rakl ngkutl = $2 \times 8 + 2$
7	pombingkutl rakl or kotraktika	19	ki rakl raktika ngkutl = $2 \times 8 + 3$
8	engkaka or ki dende = one 8	20	ki rakl tembokaka = $2 \times 8 + 4$
9	pombi ti ngkutl = $2 \times 4 + 1$	21	ki rakl pombingkutl = $2 \times 8 + 5$
10	pombinrakl ngkutl = $2 \times 4 + 2$	22	ki rakl pombingkutl dende = $2 \times 8 + 6$
11	pombinraktikangkutl = $2 \times 4 + 3$	23	ki rakl pombingkutl rakl = $2 \times 8 + 7$
12	tembokakapoket = three 4s	24	ki raktika three 8s

Based on Melpa dialect: Vicedom and Tischner, 1948, Vol. 1, pp. 237-8 (cited in Lean, 1988, Vol. 9, p. 27.)

Some Papua New Guinean language groups use names for eight and nine that subtract two and one from 10 rather than add on to five. In several groups, the counting matches the naming of parts of the body. That is body tally systems tally numbers by body parts from left small finger up the arm, across the head, and down the other arm. In this case, a *man* is reached, for example, when 23 or 27 or 59 tallies have been marked off against the different body parts. The odd number comes because of the central point (e.g. nose) on the head.

4.4 Laws for Number Systems

Number systems follow particular laws. For example, we can change the order of numbers when we multiply or add real numbers. This exchange of places (like a commuter travelling between work and home) is called the commutative law. We are quite familiar with it.

If x and y are real numbers, then $x \times y = y \times x$

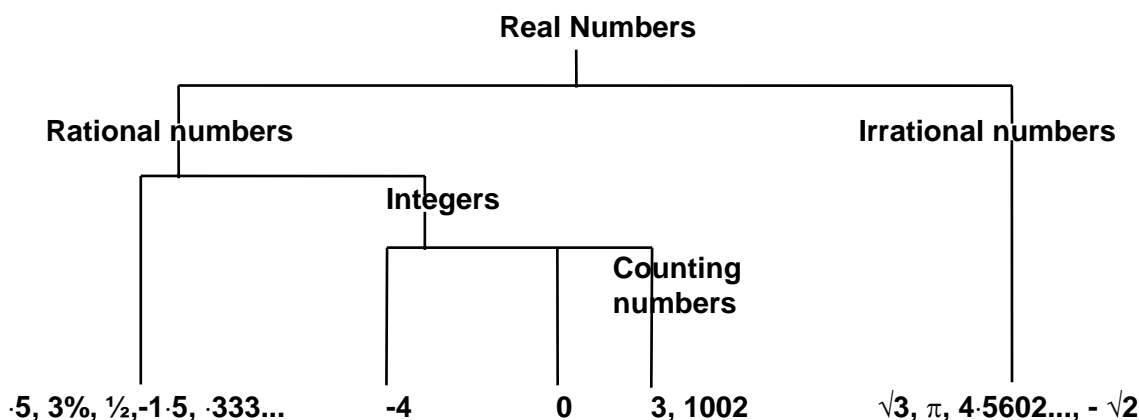
The result is a real number.

However, if we subtract counting numbers we do not necessarily get a counting number (e.g. $3 - 7$). Nor can we exchange the 3 and 7 and get the same number.

Most people learn these number laws incidentally in school. However, some children find them difficult because of the lack of recognition of the importance of reading left to right and the order of position of the numbers.

4.5 Relating Types of Numbers

It helps to have an idea of how different types of numbers can be related. Again our naming and grouping are socially decided.



This diagram is a typical representation. It has many hidden meanings. First we need to know what each of the words mean. Secondly, we need to recognise that any group branching from the one above has numbers equal to numbers in the larger group. Branches give all the groups possible for the large group.

Rational numbers represented by decimal numeration will be

- repeating decimals (e.g., $4.325325325\dots$) or
- terminating decimals (e.g., 27.37).

Any real number that is not rational is irrational so these decimals are non-terminating and non-repeating. Sometimes special short-hand symbols are used like the square root sign ($\sqrt{\quad}$). Note that some surds ($\sqrt{4}$) are rational while others ($\sqrt{5}$) are irrational.

Thinking Task 4.1

- If you divide a counting number by a counting number do you always get a counting number?

- Counting numbers are equal to whole positive integers in the rational number set. Does division by whole positive integers give a rational number?
- Try two divided by nine. Represent this as a vulgar fraction and a decimal fraction. What kind of decimal is it?
- Is this a form for a rational number?
- Can you find a pattern for $2 \div 99$, $2 \div 999$?
- How about $2 \div 90$, $2 \div 900$? Can you predict before you try?

4.6 Place Value

This is one of the most difficult aspects of our numeration system. It has two main concepts:

- grouping unit or composite unit, and
- position of a number gives its value

Thinking Task 4.2

Imagine that you are helpers in the lolly factory. First the factory **bundles** the lollies into packs of 3. (If you have a bundle of multilink cubes, you can do this.) Then the factory bundles 3 packs into a box. Then the factory bundles 3 boxes into a carton.

- If you are in a class group, get every group to record the number of cubes they have.
- How could you work out a way of recording the different amounts in a table form?

(A common suggestion is the idea of headings of cartons, boxes, packs, units with amounts recorded in columns underneath.)

Note that the position or place of the column tells you how big the size of the number underneath is.

Although we need not emphasise the base being used, we do need to understand the idea of bundling to get a grouping or **composite unit**. In the above scenario, the grouping unit was three as a unit.

- Do the same kind of activity with paddlepop stick bundles or sticks of 10 multilink blocks. This time the group unit is ten as a unit.

In our place value system, **zero has two roles**. It can tell us that there **are** no tens or other type of composite unit and **it holds the place** for the hundreds or other column numbers.

For example, in 2 305, zero tells us there are no tens and it holds the place so the 2 and 3 are clearly in the thousands and hundreds column.

The position of a number tells the type of composite unit that is being used. Is it tens, hundreds, or thousands? Hundred is a *composite of ten* tens, following the same pattern of ten being a *composite of ten* ones.

In our place value system the *ten* to make a new composite unit has many advantages. The cycles consistently use the same base of 10 unlike some of the traditional society systems like Ndom which switched over from 6 to 36 or the systems that have cycles that switch from 5 to 20. One of the advantages of a consistent base ten is the ease of adding in columns or setting up other algorithms for calculating.

These hidden messages of the place value system are complicated for some people by the difficulty of spatial arrangements. For some people the relative position on a page is not an easy focus. Some people find it hard to distinguish direction.

A second difficulty is the dominance of the size of the digit (face value). The sense of size needs emphasising in teaching. It is hard to picture just how big 1 000 000 is. We need a lot of dots packed closely together to even see it physically.

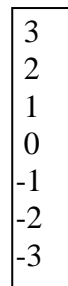
Chapter 5

More about Numbers

5.1 Negative Numbers

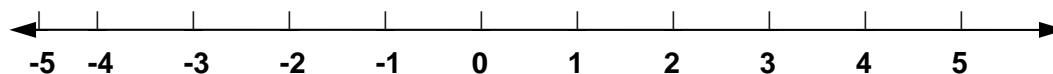
What do we mean by negative numbers? How might these be represented?

The easiest understanding of negative numbers is in terms of opposite direction. When the temperature falls below zero, it gets colder and colder. The lower it gets the larger the digits but in fact the smaller the number. It is further from zero.



If you go below ground into the car parks, carpark floors are usually labelled as 1, 2, 3 etc floors below ground level. We could say the lowest floor is -3.

On a number line, we represent negative numbers to the left of zero. The further from zero, the larger the absolute value (that is the number without its sign) but the smaller the number. For example, -235 is smaller than -78 is smaller than -15.



We say that -12 is opposite 12 because both numbers are the same distance from zero but in opposite directions.

5.2 Addition with Positive and Negative Numbers

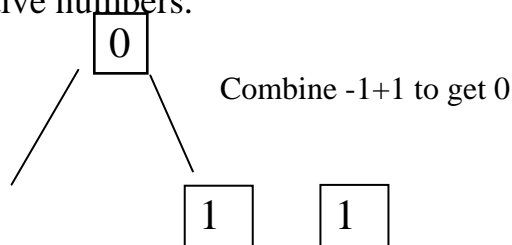
For this section, you should make up a set of cards with -1 and 1. (A sheet is attached which you can cut up for this purpose.)

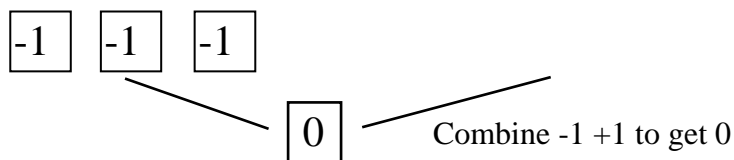
We can also think of -3 as being 3 lots of -1. This makes sense of addition problems. For example,

$$\begin{array}{c}
 \boxed{-1} \quad \boxed{-1} \quad \boxed{-1} \\
 -3
 \end{array}
 +
 \begin{array}{c}
 \boxed{-1} \quad \boxed{-1} \\
 -2
 \end{array}
 \text{ is } -5$$

$$\text{If } \boxed{-1} + \boxed{+1} = \boxed{0}$$

then we can add negative and positive numbers.





For example,

$$\begin{aligned}
 & -3 \quad + \quad 2 \\
 & = \text{two zeroes} + -1 \\
 & -3 + 2 = -1
 \end{aligned}$$

You notice it is the difference between the absolute values, 3 and 2. (Absolute value is the size without considering the sign or direction).

The larger absolute provides the sign.

The order for addition is not important. For example, $2 + -3 = -1$

Thinking Task 5.3

Use cards with -1 and 1 to calculate a range of addition with positive and negative numbers. Record and summarise the solution patterns.

5.3 Subtraction with Positive and Negative Numbers

If we use our cards again, we can easily subtract, for example, $-5 - -3$.

- Place out 5 $\boxed{-1}$ cards and take away 3 of the $\boxed{-1}$ cards.

We have left 2 of the $\boxed{-1}$ cards.

$$-5 - -3 = -2.$$

What is the pattern this time?

What happens if we want to subtract positive numbers?

We can use dummy zeroes. That is we add enough “zeroes” $\boxed{-1} \boxed{1}$ to have positives to take away.

For example, $-5 - 3$

$\boxed{-1}$ $\boxed{-1}$ $\boxed{-1}$ $\boxed{-1}$ $\boxed{-1}$
 add pairs of $\boxed{-1}$ $\boxed{1}$
 take away the 3 positives

You are left with 8 $\boxed{-1}$

That is, $-5 - 3 = -8$

We can also find $3 - -5$ gives a result of 8 by adding pairs of “zeroes” so we have 5 $\boxed{-1}$ cards to take away. We are left with 8 $\boxed{1}$

Thinking Task 5.4

- Try other subtractions and record the pattern.

5.4 Multiplication and Division of Negative Numbers

Initially we can think of a similar situation. 3 groups of -2 is like 3 groups of 2 $\boxed{-1}$ cards. We have a total of 6 $\boxed{-1}$ cards.

$$3 \times -2 = -6$$

We can apply the commutative laws. $-2 \times 3 = -6$

Thinking Task 5.5

Use cards to make groups and record results.

$$5 \times -2 =$$

$$5 \times 3 =$$

$$-3 \times 4 =$$

$$3 \times -3 =$$

What is the general pattern?

For *division*, we can divide up 6 $\boxed{-1}$ cards into 3 groups.

To record this we get

$$-6 \div 3 = -2$$

We can also interpret $-6 \div -2$ as making groups of -2 and we will get 3 groups.

So $-6 \div -2 = 3$

Thinking Task 5.6

Use cards to share or group.

$$-8 \div 4$$

$$-6 \div -3$$

$$-9 \div 3$$

$$-12 \div 4$$

$$-12 \div -3$$

Comment

However, the other problems with division and multiplication are not so easy to represent with our cards.

Thinking Task 5.7

Try some questions on the calculator and decide on the patterns that are occurring. The calculator key for getting a negative number is +/-

$$-3 \times -4 =$$

$$-2 \times -5 =$$

$$10 \div -2 =$$

$$12 \div -3 =$$

You might see these as opposites of other questions. For example,

$$3 \times -4 = -12$$

The opposite of -3×-4 is the opposite of $-12 = -(-12) = 12$.

Chapter 6

Power Functions

6.1 Index Notation

Four words in mathematics are synonymous. They are the terms index, power, exponent, and logarithm (often called log). Many students have trouble with these words. In particular, they have difficulty with the idea of log because it is often used in function form.

Index notation is expressed like 5^3 . The meaning of this is $5 \times 5 \times 5$. That is 5 multiplied by itself 3 times. The numbers in index notation grow quickly.

$$5 \rightarrow 25 \rightarrow 125$$

Thinking Task 6.1

- About how big do you think 2^{10} might be?
- Use a calculator to check.

In a primary school calculator, you can use its constant function system of $2 \times = =$ (do this 10 times). Watch how the numbers grow.

In a scientific calculator, you can use a power key e.g. $2 \boxed{x^y} 10 =$

For the number $5^3 = 125$, we say 3 is the *index* or *power* or *log*.

- What is the log of 1024 to base 2?

The answer is 10. We say our sentences slightly differently for each word.

- What is the power of 2 that gives 1024?

In a shorthand way, we can say log of 125 relative to 5 is 3, that is

$$\log_5 125 = 3$$

- Write the shorthand for log of 1024 relative to 2.

This means the chances are that we can use numbers to calculate with indices. If we divided 5^7 by 5^7 we can subtract the indices giving us zero. However we also know that a number divided by itself is also one. So we have a special relationship.

$$7^0 = 1$$

Thinking Task 6.2

- Does it matter what base (in this case 7) we have?
- Will any number's zero power always be 1?

Have you also noticed that we have used the word *base* in two different ways. Once we used it to describe the grouping or composite unit in a counting system. For example, we have a base 10 system. The Ndom had a base 6 or a system that included a 6-cycle.

We have now used the word base to refer to the number that is given a power or log or index.

6.5 Using Other Numbers as Indices

Interestingly when we have index notation, we might divide numbers that give us negative numbers.

For example, $5^3 \div 5^7$

- What might this number be in index form?

$$5^3 \div 5^7 = 5^{3-7} = 5^{-4}$$

However, what does this mean. Go back to the other representation

$$\frac{5 \times 5 \times 5}{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5} = \frac{5^3}{5^7}$$

$$\frac{1}{5 \times 5 \times 5 \times 5} = \frac{1}{5^4}$$

If this is the case,

$$5^{-4} = \frac{1}{5^4}$$

We can use these different ways of representing powers. The indices can be manipulated like ordinary numbers.

Thinking Task 6.3

Simplify $3^5 \times 3^4 \div 3^2$

Write it out in a long way and simplify by making $\frac{3}{3}$ equal to 1

Does your answer make sense?

Tackle a question with negative indices.

e.g. $3^{-2} \times 3^4 \times 3^5 \div 3^3$

Check it another way.

What do you think $\frac{1}{3^{-2}}$ means? Write it another way.

We cannot leave the indices until we know what to do with powers of powers. For example, $(5^3)^2$ This means $5^3 \times 5^3 = 5^{3 \times 2} = 5^6$

Fun. Just as we could multiply for repeated groups, we can multiply for indices if the numbers are multiplied.

We should also note that all the time we have had powers of the same number. We can only apply the index rules if the numbers have the same base. That is we cannot apply the laws to $3^2 \times 5^3$. Why?

Thinking Tasks 6.4

Simplify the numbers you can simplify.

$$2^3 \times 2^5$$

$$2^3 \times 2^{-3}$$

$$2^3 \times 2^{-5}$$

$$2^3 \times 5^3$$

$$2^3 \times 3^4$$

- What do you think of $2^3 \times 5^3$?
- Can the commutative law of multiplication help to simplify as you have the same number of 2s and 5s in this product.

What might a fractional index mean? For example $3^{1/2}$

This refers to $\sqrt{3}$

It means we can simplify other expressions.

For example $\sqrt{5^6}$
 $(5^6)^{1/2}$

- Multiply the indices $5^{6 \times 1/2} = 5^3$

Thinking Task 6.5

- Show what $\sqrt{5^6}$ might mean if written in longhand and why we could simplify as above.
- How do we write $\sqrt[3]{15}$ in fraction index notation?
- Calculate $\sqrt[3]{125}$
- Can we calculate $\sqrt[3]{-125}$? Why or why not?

- What about $\sqrt{-16}$? Why or why not?

6.6 Large and Small Numbers

How do we name very large numbers?

Here are a few.

1 000 000	million
1 000 000 000	billion
1 000 000 000 000	trillion
10^{100}	googol

Americans and newspapers sometimes refer to a billion as 10 million.

Thinking Task 6.6

Write the numbers immediately above as powers of 10.

As science progressed, there was a need to represent large and small numbers efficiently. For this reason, the system called scientific notation came into vogue. It is a system that tells you which column the unit focus is on.

For example, 3×10^5 tells you immediately that the 3 is in the *fifth* column further over in ordinary notation. Five zeros are needed to hold its place.

HT abbreviates Hundred thousands

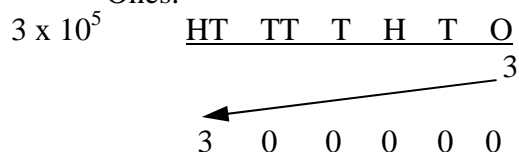
TT Ten thousands,

T Thousands,

H Hundreds,

T Tens,

O Ones.



For the number 7.254×10^3

<u>T</u>	<u>H</u>	<u>T</u>	<u>O</u>	
			7.254	
		7		
	7	2	5	4.

The 7 is in the third place and the other digits fill in the columns in order so that the relationship of 10 times the previous column is kept.

For the number 3.257×10^{-3}
the 3 is a digit in the third column to the right, it is a number less than 1.

O abbreviates ones,

Tth is tenths,

Hth is hundredths,

Tth is thousandths,

TTth is ten thousandths,

HTth is hundred thousands,

Mth is millionths.

<u>O</u>	<u>Tth</u>	<u>Hth</u>	<u>Tth</u>	<u>TTth</u>	<u>HTth</u>	<u>Mth</u>	
	3.	2	5	7			
				3	2	5	7
0.	0	0	3	2	5	7	

Scientific notation emphasises the size of a number by the power of 10. It is possible to see how close numbers are by the power of ten.

When numbers have the same power, it may be the size of the digits that counts.

It is noteworthy that the system of powers is so useful yet closely linked with the base 10 column system.

Nevertheless, we can still carry out calculations in this notation and it provides a means of entering large and small numbers into a calculator. Use the EXP key to enter the numbers. This function is completed before any addition or subtraction. Because of the commutative law, we cannot easily falter if you did enter the scientific notation with a multiplication key but forgetting to use the EXP key with a division can cause a problem.

Thinking Task 6.7

Estimate the answer to the following question. Think about the powers of 10 in particular.

$$2.3 \times 10^5 \div 5.4 \times 10^3$$

Enter the following into a scientific calculator using the \boxed{x} and $\boxed{x^y}$ keys

Then try it using the EXP key.

- Which one is correct?

Now this is a matter of reading the hidden messages in our shorthand mathematics. If you did not know about scientific notation, you would have to put brackets around 5.4×10^3 .

Comments on Thinking Tasks

- 1.2 * see ideas on p. 9
- 2.1 * other examples could include $10 - 7 = 3$
 * $7 + 3 = 10$ so $7 + 6 = 7 + 3 + 3 = 10 + 3 = 13$ or $7 + 6$ is double 6 plus one or $12 + 1 = 13$
- 2.2 * 10 metre sticks are used where as 48 is less than half a metre.
 The place value makes a huge difference in the size represented by the digit. The sticks visually show the size better but they are cumbersome to use.
- 2.3 * task about the goat. Each of the areas is a part of a circle. The part is a semi-circle unless the rope wraps around the end of the building and then quadrants are created unless the shape of the building is not rectangular. The rope when taut forms a straight line. The locus is a part of a circle each time.
- 3.1 * A numeral signifies a number which is an abstract concept like threeness. The numeral on a number line is representing the length from the origin to the numeral marker. The length for 5 is longer than for 3. If each length to 3 and 5 were made separately and placed side-by-side, the length would be emphasised over the position of the numeral. Numbers represented to the right are larger. Each numeral is placed at an equal distance or equally spaced so that the visual representation is meaningful and so that comparisons can be adequately made.
- * Some scales can be different. For example, logarithm scales used on old-fashioned slide rulers or to fit information that grows exponentially onto the same graph.
 - * Scales need not be integers. They numerals and marking could be a multiple of 1. Frequently the scales are multiples of 2 or 10 or 100 or divisible by 10 or 100.
 - * The number line can be used for addition by moving to the right where as multiplication would be jumping with the same big strides along the line. Subtraction as the reverse of addition means moving to the left whereas division would consist of dividing up the length into equal parts.
 - * There are points on the number line representing off the different kinds of real numbers that are mentioned.
- 3.2 * Divide both sides by 3
 Remove the parentheses as these are not needed
 Take 5 from both sides
 Do the subtraction of numerals

*The parentheses could be multiplied out first giving $3x + 15 = 18$

Subtract 15 from both sides leaving $3x = 3$

Divide both sides by 3 leaving $x = 1$

3.3 $* = 4 \times 2$

$$= 8$$

or $4 \times 5 - 4 \times 3$

$$= 20 - 12$$

$$= 8$$

$$* = 2x \times 3x - 2x \times 5$$

$$= 6x^2 - 10x$$

3.4 *See p.17 for an example.

4.1 *Not all divisions result in a whole number; most are fractions or rational numbers like $\frac{8}{3}$

$$\frac{2}{9} \text{ is } 0.222\dots; \frac{2}{99} \text{ is } 0.020202\dots; \frac{2}{999} \text{ is } 0.002002002\dots; \frac{2}{90} \text{ is } 0.0222\dots$$

4.2 * carton box pack unit

$$1 \quad 2 \quad 1 \quad 1$$

5.3 * $5 + -3$ is 1 1 1 1 1 + -1 -1 -1 \rightarrow 1 1 since the other 1s and -1s make zeroes. The result is positive as 5 is larger than 3.

$-5 + -3$ gives eight -1s. Result is -8 as we get more -1s.

5.4 * $-8 - -5$ leaves three -1s. It is the difference between 8 and 5.

$-5 - -8$ we don't have enough -1s so need to add 3 dummy ones so we get left with 3 1s after taking away all 8 -1s. 3 is the difference between 5 and 8

5.5 * 5 groups of -2 gives -10; 4 groups of -3 is -12; 5 groups of 3 is 15; 3 groups of -3 is -9.

5.6 * -2 in each of the four groups; 2 groups of -3; -3 in each of the 3 groups; -3 in each of the four groups.

5.7 *Negative of the groups of negative numbers ends up as positive due to the counteraction of the two negatives. Dividing up a positive into negative groups has to be seen as the opposite of negatives so the answers are $10 \div -2$ is -5; $12 \div -3$ is -4

6.1 * 2^{10} is about 1 000

6.2 * $\frac{3 \times 3 \times 3 \times 3}{3 \times 3 \times 3 \times 3} = \frac{3^4}{3^4} = 3^{4-4} = 3^0$ All real numbers.

6.3 * $3^{5+4-2} = 3^{9-2} = 3^7$

$$\frac{3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = 3^7$$

6.4 * $3^{-2} \times 3^4 \times 3^5 \div 3^3 = 3^{-2+4+5-3} = 3^{2+5-3} = 3^{7-3} = 3^4$

$$\frac{1 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3} = 3 \times 3 \times 3 \times 3 \text{ or } 3^{9-5} = 3^4$$

- 6.4 * 2^8 ; $2^0 = 1$; 2^{-2} ; cannot simplify by indices rules as 2 and 5 are different bases but as they are to the same power $2 \times 5 = 10$ so 10^3 ; cannot simplify as 2 and 3 are different bases.
- 6.5 * $\sqrt{5 \times 5 \times 5 \times 5 \times 5 \times 5}$ and each square root of a pair of 5s is 5 so we have $5 \times 5 \times 5$ or 5^3 . $15^{1/3}$, 15 power of a third - it is an irrational number; $-5 \times -5 \times -5 = (-5)^3 = -125$ so $\sqrt[3]{-125} = -5$. It is the inverse or undoing operation. We cannot do square root of -16 as no square is negative.
- 6.6 * 10^6 ; 10^9 ; 10^{12} ; 10^{10}
- 6.7 * $2 \div 5$ is about a half. 0.5 and $10^{5-2} = 10^2$. Estimate as about 50.
EXP key gives 42.59; the other keys, if parentheses are not used, will be wrong because the last product is actually to be divided.

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+1	+1	+1	+1	+1
+1	+1	+1	+1	+1
-1	-1	-1	-1	-1
-1	-1	-1	-1	-1
0	0	0	0	0
+1	+1	+1	+1	+1
-1	-1	-1	-1	-1

